Stability of Classical Solutions of Nonlocal Gravity

非局部引力经典解的稳定性

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摘要

In this section we discuss the stability of classical solutions of nonlocal gravity. We consider a special class of nonlocal theories that admits all the maximally symmetric and Ricci-flat vacuum solutions of general relativity with and without cosmological constant, and show that such solutions are as stable as in Einstein's theory. This implies the stability of the Minkowski spacetime, black hole solutions and gravitational waves.

在本节中我们讨论非局部引力经典解的稳定性。我们研究一类特殊的非局部引力理论，这类理论允许广义相对论中所有带有宇宙学常数和不带有宇宙学常数的极大对称真空解与里奇平坦真空解，我们证明了这类解的稳定性与爱因斯坦引力理论中的结果一致。这意味着闵氏时空、黑洞解与引力波都是稳定的。

Keywords

关键词

Nonlocal gravity - Stability of classical solutions

非局域引力 - 经典解的稳定性

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Introduction

引言

The study and characterization of classical solutions of nonlocal gravitational theories is an important task; see [1] for a review of nonlocal gravity. In fact, although nonlocal models are designed to find a way out to the quantum-gravity problem, so that one expects them to be relevant in the quantum-gravity regime, one has to pay attention to the properties of classical solutions, which describe the universe from cosmological to solar system scales.

非局部引力理论经典解的研究与表征是一项重要课题；关于非局部引力的综述可参见文献 [1]。事实上，尽管非局部模型是为解决量子引力问题而构建的——因此人们预期它们在量子引力区间才发挥主要作用，但我们仍必须关注经典解的性质，正是这些经典解描述了从宇宙学到太阳系尺度的宇宙。

Indeed, nonlocal gravity must be renormalizable at quantum level, and it must also have classical solutions that agree with astrophysical and cosmological tests of gravitation. Since Einstein gravity is quite successful as classical model of gravitational interactions, one way of proceeding to formulate a viable nonlocal quantum gravitational model is to require that the new theory must encompass all the relevant classical solutions of general relativity, as black holes and gravitational waves among others. What is more, these solutions must be stable.

实际上，非局部引力不仅需要在量子层面可重整化，还必须拥有满足引力的天体物理和宇宙学观测检验的经典解。由于爱因斯坦引力作为引力相互作用的经典模型十分成功，构建可行非局部量子引力模型的一种方法是: 要求新理论包含广义相对论所有相关的经典解，比如黑洞、引力波等等。此外，这些解还必须是稳定的。

For instance, one can build nonlocal gravitational models in such a way that all the vacuum solutions of general relativity without a cosmological constant, i.e., Ricci-flat spacetimes, are also solutions of the nonlocal theory. In doing so, one automatically includes the Minkowski spacetime, gravitational waves and black hole solutions. This is the case of the following minimal nonlocal action

例如，我们可以构建非局部引力模型，使得广义相对论中所有不带宇宙学常数的真空解，即里奇平坦时空，同时也是该非局部理论的解。按照这种方式，闵氏时空、引力波和黑洞解会被自动纳入理论。以下这个最小非局部作用量就是这种情况:

$$S_{\mathrm{g}} = -\frac{2}{\kappa_D^2} \int d^4x \sqrt{-g} \left[R + G_{\mu\nu} \gamma(\Box) R^{\mu\nu} + V(\mathcal{R}) \right]. \tag{1}$$

where $R, R_{\mu\nu}$, and $G_{\mu\nu}$ are the Ricci scalar, Ricci curvature, and the Einstein tensor, respectively. This action includes a generalized potential $V(\mathcal{R})$ which must be at least cubic in the Ricci scalar and Ricci tensor for our purposes. In this notation, $\mathcal{R}$ stays for Ricci scalar, Ricci tensor, and their covariant derivatives. The nonlocality is encoded in the form factor $\gamma(\Box)$, which ensures the quantum renormalizability and unitarity of the model, as discussed elsewhere in this book; see also [1, 2]. This operator depends on the non locality length scale $\ell \equiv \sqrt{\sigma}$ and the covariant d’ Alembertian $\Box \equiv \nabla^\mu \nabla_\mu$, and it is defined as

其中 $R, R_{\mu\nu}$ 、和 $G_{\mu\nu}$ 分别是里奇标量、里奇曲率和爱因斯坦张量。该作用量包含了一个广义势 $V(\mathcal{R})$ ，就我们的研究而言，该势在里奇标量和里奇张量中至少是三次的。按照此处的记号，$\mathcal{R}$ 代表里奇标量、里奇张量及其协变导数。非局域性由形状因子 $\gamma(\Box)$ 编码，正如本书其他部分讨论的，该形状因子保证了模型的量子可重整化性和幺正性，另可参见 [1, 2]。该算符依赖于非局域长度标度 $\ell \equiv \sqrt{\sigma}$ 和协变达朗贝尔算符 $\Box \equiv \nabla^\mu \nabla_\mu$ ，其定义为:

$$\gamma(\Box) \equiv \frac{f(\sigma\Box) - 1}{\Box}, \tag{2}$$

where $f(z)$ is an entire analytic function without zeros at finite complex z, e.g., $f(z) = \exp[H(z)]$ with polynomial $H(z)$. This requirement on the form of $f(z)$ ensures that the theory is unitary; see [2].

其中 $f(z)$ 是在有限复 z 上没有零点的整解析函数，例如 $f(z) = \exp[H(z)]$ 配多项式 $H(z)$。对 $f(z)$ 形式的这一要求保证了理论的幺正性，参见文献 [2]。

The classical equations of motion, which are the nonlocal generalization of the Einstein equations, are obtained by variation of the action (1) with respect to the metric tensor $g_{\mu\nu}$ and read

经典运动方程是爱因斯坦方程的非局部推广，它通过对作用量 (1) 关于度量张量 $g_{\mu\nu}$ 变分得到，形式为:

$$f(\sigma\Box) G_{\mu\upsilon} + \left(g_{\mu\upsilon}\nabla_\alpha\nabla_\beta - g_{\alpha\mu}\nabla_\beta\nabla_\upsilon\right)\gamma(\Box) G^{\alpha\beta} + Q_{2\mu\upsilon}(\mathrm{Ric}) = \frac{8\pi G_\mathrm{N}}{c^4} T_{\mu\upsilon}, \tag{3}$$

where $T_{\mu\upsilon} \equiv -\left(2/\sqrt{-g}\right)\delta S_m/\delta g_{\mu\upsilon}$ is the matter stress-energy tensor, c is the speed of light, and G_N is the Newtonian constant of gravitation. For Ric we mean Ricci scalar and Ricci tensor, while Q_2 (Ric) is a sum of terms at least quadratic in Ric (this assumption will be crucial in what follows), e.g.,

其中 $T_{\mu\upsilon} \equiv -\left(2/\sqrt{-g}\right)\delta S_m/\delta g_{\mu\upsilon}$ 是物质能动张量，c 是光速，G_N 是牛顿引力常数。我们用 Ric 指代里奇标量和里奇张量，而 Q_2 (Ric) 是 Ric 中至少二次项的和，下文将看到这一假设至关重要，例如:

$$\sigma\left((\sigma\Box)^n R_{\mu\alpha}\right)\left((\sigma\Box)^m R^\alpha{}_\upsilon\right) \quad \text{or} \quad \sigma^2\left((\sigma\Box)^n R_{\mu\alpha}\right)\left((\sigma\Box)^m R^\alpha{}_\upsilon\right)\left((\sigma\Box)^l R\right),$$

(4)

for integer n, m, l.

其中 n, m, l 为整数。

We omit the derivation of equation (3), which can be found in Reference [3]; see also [4] for the formulation of the nonlocal theory ”a la Palatini”. However, it is straightforward to recognize that Ricci-flat solutions of the Einstein equations in vacuum, namely spacetime metrics satisfying the equation $R_{\mu\nu} = 0$, are also solutions of the modified Einstein equations (3) in vacuum, i.e., for $T_{\mu\nu} = 0$. These include the Minkowski metric, the Schwarzschild and Kerr black hole solutions, and gravitational waves among others.

我们省略了方程 (3) 的推导，推导可在参考文献 [3] 中找到；关于“帕拉蒂尼式”非局部理论的构建可参见文献 [4]。但我们可以很容易看出，真空中爱因斯坦方程的里奇平坦解，即满足方程 $R_{\mu\nu} = 0$ 的时空度量，同时也是真空中修正后爱因斯坦方程 (3) 的解，也就是对应 $T_{\mu\nu} = 0$ 时的解。这些解包括闵氏度量、史瓦西和克尔黑洞解以及引力波等等。

The question whether classical solutions are stable or not in the framework of nonlocal gravity arises naturally, since only stable solutions have physical relevance. Such a question is hard to answer for the general class of nonlocal theories of gravity. However, in some special cases it can be proved that some of the solutions of the nonlocal modified Einstein equations are stable. Fortunately, these turn out to include the relevant solutions of general relativity, that is, Minkowski spacetime, black holes, and gravitational waves.

在非局部引力框架下，经典解是否稳定的问题自然而然地浮现，因为只有稳定的解才具有物理意义。对于一般类别的非局部引力理论，这个问题很难回答。不过，在一些特殊情形下，可以证明非局部修正爱因斯坦方程的部分解是稳定的。幸运的是，这些稳定解恰好包含广义相对论的相关解，即闵氏时空、黑洞和引力波。

For instance, for the model defined by the Lagrangian (1), one can prove that the Minkowski spacetime is stable under small perturbations. This also implies the stability of gravitational waves. However, it is very difficult to say something about the stability of other Ricci-flat solutions, e.g., black holes. To circumvent this difficulty, one can generalize the nonlocal action (1) as in (35). The new theory, which includes a cosmological constant Λ , has all the Ricci flat (for $\Lambda = 0$) and maximally symmetric (for $\Lambda \neq 0$) solutions of the Einstein equations in vacuum. Moreover, such spacetimes are as stable as in general relativity. Indeed, if a maximally symmetric or Ricci-flat solution of the Einstein's theory (with or without a cosmological constant) is stable, it is also a stable solution of the nonlocal theory (35). This implies, for instance, that black hole solutions are stable in the nonlocal model (35).

例如，对于由拉格朗日量 (1) 定义的模型，我们可以证明闵氏时空在小扰动下是稳定的，这也意味着引力波的稳定性。但对于其他里奇平坦解 (例如黑洞)，我们很难给出关于其稳定性的结论。为解决这一难题，我们可以像式 (35) 那样推广非局部作用量 (1)。这个包含宇宙学常数 Λ 的新理论，囊括了真空爱因斯坦方程所有的解: $\Lambda = 0$ 对应的里奇平坦解和 $\Lambda \neq 0$ 对应的最大对称解。此外，这些时空的稳定性和广义相对论中的情况一致。事实上，如果爱因斯坦理论 (带或不带宇宙学常数) 的某个最大对称解或里奇平坦解是稳定的，那么它也必然是非局部理论 (35) 的稳定解。例如，这就意味着黑洞解在非局部模型 (35) 中是稳定的。

To give an explicit example of how these results can be proved, in the following sections we will show that the Minkowski spacetime is a stable solution of the model (1). Indeed, in section ”Stability of the Minkowski Spacetime in General Relativity” we will review the stability theorem for the Minkowski spacetime in general relativity. Then, in section ”Stability of the Minkowski Spacetime in Nonlocal Gravity” we will generalize this theorem to the case of nonlocal gravity (1). To conclude, in section ”Generalization to Ricci-Flat and Maximally Symmetric Solutions” we will discuss the stability of Ricci flat and maximally symmetric solutions of the nonlocal model (35).

为了给出这些结果证明过程的具体示例，我们将在后续章节中证明闵氏时空是模型 (1) 的稳定解。我们会先在“广义相对论中闵氏时空的稳定性”一节回顾广义相对论中闵氏时空的稳定性定理，再在“非局部引力中闵氏时空的稳定性”一节将该定理推广到非局部引力 (1) 的情况。最后，我们会在“推广至里奇平坦解与最大对称解”一节讨论非局部模型 (35) 中里奇平坦解和最大对称解的稳定性。

Stability of the Minkowski Spacetime in General Relativity

广义相对论中闵氏时空的稳定性

The stability of the Minkowski spacetime in general relativity has been established long time ago [5,6]. In this section we will review this result, introducing the notions of Strongly Asymptotically Flat (SAF) condition and Global Smallness Assumption (GSA) of initial data sets, and stating the stability theorem.

广义相对论中闵氏时空的稳定性早已得到证明 [5,6]。本节我们将回顾这一结果，引入强渐近平坦 (SAF) 条件和初始数据集全局小假设 (GSA) 的概念，并陈述稳定性定理。

Let us consider a globally defined differentiable function τ whose gradient $D\tau$ is everywhere time-like. We call τ a time function. The level surfaces of τ give a foliation of the spacetime $\sum_\tau$. Indeed, the spacetime can be parameterized by the points of a slice $\sum_{\tau_0}$ by following the integral curves of $D\tau$, so that it is diffeomorphic to the product manifold $\mathbb{R}\times\sum_{\tau_0}$. In the coordinate system (τ,ξ) constructed in this way, where ξ are coordinates over the slice $\sum_{\tau_0}$, the metric takes the form

考虑一个全局定义的可微函数 τ ，其梯度 $D\tau$ 处处类时。我们称 τ 为类时函数，τ 的等值面对时空 $\sum_\tau$ 给出一个叶状结构。事实上，我们可以沿着 $D\tau$ 的积分曲线，用切片 $\sum_{\tau_0}$ 上的点参数化整个时空，因此时空微分同胚于乘积流形 $\mathbb{R}\times\sum_{\tau_0}$ 。在这样构造的坐标系 (τ,ξ) 中，其中 ξ 是切片 $\sum_{\tau_0}$ 上的坐标，度量可写为如下形式

$$ds^2=\phi^2(\tau,\xi)\,d\tau^2+\chi_{ij}(\tau,\xi)\,d\xi^i d\xi^j, \tag{5}$$

with $\chi_{ij}(\tau,\xi)$ a 3×3 Riemannian metric. We also define the following quantity, corresponding to the extrinsic curvature of the leaves $\sum_\tau$ as

其中 $\chi_{ij}(\tau,\xi)$ 是一个 3×3 黎曼度量。我们还定义了如下物理量，对应叶 $\sum_\tau$ 的外曲率

$$\kappa_{ij}=(2\phi)^{-1}\partial_\tau\chi_{ij} \tag{6}$$

An initial data set for the Einstein equations is given by a set $\left(\sum_{\tau_0},\chi,\kappa\right)$ corresponding to the initial conditions at some time τ_0 . We define the class of initial data satisfying the SAF condition as follows:

爱因斯坦方程的初始数据集由 $\left(\sum_{\tau_0},\chi,\kappa\right)$ 给出，对应某一时刻 τ_0 的初始条件。我们如下定义满足强渐近平坦条件的初始数据类:

Definition 1. The initial data set $\left(\sum_{\tau_0},\chi,\kappa\right)$ satisfies the Strongly Asymptotically Flat (SAF) condition, if there is a coordinate system $\left(\xi^1,\xi^2,\xi^3\right)$ on $\sum_{\tau_0}$ such that, asymptotically for $|\xi|^2=\sum_{i=1}^{3}\left(\xi^i\right)^2\to\infty$ one has

定义 1. 初始数据集 $\left(\sum_{\tau_0},\chi,\kappa\right)$ 满足强渐近平坦 (SAF) 条件，若 $\sum_{\tau_0}$ 上存在坐标系 $\left(\xi^1,\xi^2,\xi^3\right)$ ，使得渐近地当 $|\xi|^2=\sum_{i=1}^{3}\left(\xi^i\right)^2\to\infty$ 时有

$$\chi_{ij}=\left(1+\frac{M}{\xi}\right)\delta_{ij}+o_4\left(\xi^{-3/2}\right),$$

$$\kappa_{ij}=o_3\left(\xi^{-5/2}\right), \tag{7}$$

where a function is said to be $o_m(\xi^{-n})$ if $\partial^l f(\xi) = o\left(r^{-n-l}\right)$ for $|\xi|^2 \to \infty$.

其中称一个函数是 $o_m(\xi^{-n})$，若当 $|\xi|^2 \to \infty$ 时满足 $\partial^l f(\xi) = o\left(r^{-n-l}\right)$。

In order to express the GSA, we define the following quantity:

为了表述全局小假设，我们定义如下物理量:

$$Q(\xi_0) = \sup_{\sum_{\tau_0}} \left\{ \left(d_0^2+1\right)^3 \left|R_{ij}^{(3)}\right|^2 \right\} + \int_{\sum_{\tau_0}} \sum_{l=0}^{3} \left(d_0^2+1\right)^{l+1} \left|\nabla^{(3)l}\kappa\right|^2$$
$$+ \int_{\sum_{\tau_0}} \sum_{l=0}^{3} \left(d_0^2+1\right)^{l+3} \left|\nabla^l B_{ij}\right|^2 \tag{8}$$

where $d_0(\xi) = d(\xi_0, \xi)$ is the Riemannian geodesic distance between $\xi_0 \in \sum_{\tau_0}$ and $\xi \in \sum_{\tau_0}$. Moreover, the curvature $R_{ij}^{(3)}$ and the covariant derivatives are constructed with the 3-metric χ_{ij} , and B_{ij} is the symmetric and traceless 2-tensor

其中 $d_0(\xi) = d(\xi_0, \xi)$ 是 $\xi_0 \in \sum_{\tau_0}$ 与 $\xi \in \sum_{\tau_0}$ 之间的黎曼测地线距离。此外，曲率 $R_{ij}^{(3)}$ 和协变导数由三维度量 χ_{ij} 构造得到，且 B_{ij} 是对称无迹二阶张量

$$B_{ij} = \varepsilon_j^{ab} \nabla_a^{(3)} \left(R_{ib}^{(3)} - \frac{1}{4} \chi_{ib} R^{(3)} \right). \tag{9}$$

Now we can give the following:

现在我们可以给出:

Definition 2. The metric χ_{ij} satisfies the Global Smallness Assumption (GSA), if there is a sufficiently small positive parameter μ such that

定义 2. 度量 χ_{ij} 满足全局小假设 (GSA)，若存在一个足够小的正参数 μ 使得

$$\inf_{\xi_0 \in \sum_{\tau_0}} Q(\xi_0) < \mu \tag{10}$$

The stability of the Minkowski metric in general relativity GR is stated by the following:

广义相对论 (GR) 中闵氏度量的稳定性可陈述如下:

Theorem 1. Any SAF initial data set which satisfies the GSA leads to a unique, globally hyperbolic, maximal, smooth, and geodesically complete solution of the Einstein vacuum Equations foliated by a normal, maximal time foliation. Moreover, this solution is globally asymptotically flat.

定理 1. 任意满足全局小假设 GSA 的强渐近平坦 SAF 初始数据集，都给出爱因斯坦真空方程唯一、全局双曲、极大、光滑且测地完备的解，该解可由法向极大类时叶状结构分层。此外，该解是全局渐近平坦的。

Therefore, the SAF condition and the GSA define the class of small perturbations under which the Minkowski metric is stable. We remind that a globally asymptotically flat spacetime is such that its Riemann curvature approaches to zero on any geodesic, as the corresponding affine parameter goes to infinity. It is also worth pointing out that, as a matter of fact, this is an existence theorem. In fact, it states that there exists a μ such that, if the initial data satisfies the SAF condition and the GSA for such a μ , the metric (5) has a regular evolution. However, nothing is said about the size of μ nor about how it can be estimated. Finally, we emphasize that this theorem defines the conditions under which gravitational waves are stable, as they are nothing but small perturbations of the Minkowski spacetime in vacuum.

> 因此，SAF 条件与 GSA 共同定义了闵氏度规保持稳定的小扰动类别。我们在此回顾，全局渐近平直时空满足: 当对应仿射参数趋于无穷时，其黎曼曲率沿任意测地线趋于零。同样值得指出的是，这一定理本质上是一个存在性定理。具体而言，它表明存在一个 μ ，若初始数据对该 μ 满足 SAF 条件和 GSA，则度规 (5) 存在正则演化。但该定理并未说明 μ 的大小，也未说明如何对其进行估计。最后我们要强调，该定理明确了引力波稳定存在的条件——因为引力波本身就是真空下闵氏时空的小扰动。

Stability of the Minkowski Spacetime in Nonlocal Gravity

非局部引力中闵氏时空的稳定性

We are now ready to prove that the Minkowski spacetime is a stable solution of the modified Einstein equations (3) in vacuum. Our strategy, which is based on Reference [7], consists in proving that the evolution of small perturbations of the Minkowski metric in the nonlocal theory (1) is the same as in general relativity. Thus, we conclude that such small perturbations are stable if they are stable in Einstein's theory, that is, if they satisfy the hypothesis of Theorem 1. Indeed, we can enunciate the following:

> 我们现在已经可以证明，闵氏时空是真空中修正爱因斯坦方程 (3) 的稳定解。我们基于文献 [7] 的研究策略核心在于证明: 非局部理论 (1) 中闵氏度规小扰动的演化与广义相对论中的演化完全一致。因此我们得到结论: 若小扰动在爱因斯坦理论中稳定，即满足定理 1 的假设，则这些小扰动在本文理论中也是稳定的。据此我们可以给出下述定理:

Theorem 2. Any SAF initial data set which satisfies the GSA leads to a unique, globally hyperbolic, maximal, smooth, and geodesically complete solution of the equations (3) in vacuum, foliated by a normal, maximal time foliation. Moreover, this solution is globally asymptotically flat.

> 定理 2 任意满足 GSA 假设的 SAF 初始数据集，都会给出真空中方程 (3) 的唯一、整体双曲、极大、光滑且测地完备的解，该解可由正规极大时间叶层进行分层。此外，该解是整体渐近平坦的。

To begin, we set $T_{\mu\nu} = 0$ in (3) and we consider small perturbations $h_{\mu\nu}$ of the Minkowski metric $\eta_{\mu\nu}$, i.e.,

> 首先，我们在方程 (3) 中设 $T_{\mu\nu} = 0$ ，并考虑闵氏度规 $\eta_{\mu\nu}$ 的小扰动 $h_{\mu\nu}$ ，即:

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}, \quad \text{with} \quad |\varepsilon h_{\mu\nu}| \ll 1, \tag{11}$$

where $\varepsilon \ll 1$ is a small dimensionless parameter. By means of (11) we can expand all the relevant quantities in power series in ε and solve (3) perturbatively. In such a way, we can show that the equations (3) in vacuum are verified if and only if the Einstein' s tensor $G_{\mu\nu}$ vanishes at any perturbative order in ε . That means that $g_{\mu\nu}$ in (11) is a solution of the nonlocal vacuum Einstein equations (3) if and only if it is a solution of the vacuum Einstein equations $G_{\mu\nu} = 0$. Thus, the dynamics of $h_{\mu\nu}$ in vacuum in the nonlocal model (1) is the same as in general relativity.

其中 $\varepsilon \ll 1$ 是一个无量纲小参数。利用 (11) 式我们可以将所有相关物理量按 ε 展开为幂级数，并微扰求解方程 (3)。通过该方法我们可以证明: 真空中的方程 (3) 成立，当且仅当爱因斯坦张量 $G_{\mu\nu}$ 在 ε 的任意微扰阶都为零。这意味着，(11) 式中的 $g_{\mu\nu}$ 是非局部真空爱因斯坦方程 (3) 的解，当且仅当它是真空爱因斯坦方程 $G_{\mu\nu} = 0$ 的解。因此，非局部模型 (1) 真空中 $h_{\mu\nu}$ 的动力学与广义相对论完全一致。

We start expanding $h_{\mu\nu}$ and then the Einstein' s tensor $G_{\mu\nu}$ in power series in ε , namely

我们首先对 $h_{\mu\nu}$ 做展开，之后再对爱因斯坦张量 $G_{\mu\nu}$ 按 ε 做幂级数展开，即:

$$h_{\mu\upsilon} = \sum_{n=0}^{\infty} \varepsilon^n h_{\mu\upsilon}^{(n)} \text{ and } G_{\mu\upsilon}\left(g_{\mu\upsilon}\right) = \sum_{n=1}^{\infty} \varepsilon^n G_{\mu\upsilon}^{(n)}. \tag{12}$$

The reader can find examples of perturbative expansions based on (11) in general relativity textbooks, e.g., see [8]. The convergence of (12) and other perturbative series used below, which is essential this approach, will be discussed later. Notice that the leading contribution to the Einstein tensor is of order ε because $G_{\mu\nu}^{(0)} \equiv G_{\mu\nu}(\eta) = 0$. Similarly, we can expand the covariant derivative as

读者可以在广义相对论教材中找到基于 (11) 式的微扰展开示例，例如参见文献 [8]。(12) 式及下文用到的其他微扰级数的收敛性是该方法的核心，我们会在后文讨论。注意，由于 $G_{\mu\nu}^{(0)} \equiv G_{\mu\nu}(\eta) = 0$ ，爱因斯坦张量的领头贡献为 ε 阶。同理，我们可以将协变导数展开为:

$$\nabla_\alpha = \sum_{n=0}^{\infty} \varepsilon^n \nabla_\alpha^{(n)} = \partial_\alpha + \sum_{n=1}^{\infty} \varepsilon^n \nabla_\alpha^{(n)}, \tag{13}$$

where we have shown explicitly only the first term $\nabla_\alpha^{(0)} = \partial_\alpha$, since the other terms will not be needed. We can define and expand the following differential operator that appears in (3)

由于后续推导不需要其他项，我们这里仅显式写出第一项 $\nabla_\alpha^{(0)} = \partial_\alpha$ 。我们可以对 (3) 式中出现的如下微分算子进行定义和展开:

$$\begin{aligned} A_{\mu\upsilon\alpha\beta} &\equiv \left(g_{\mu\upsilon}\nabla_\alpha\nabla_\beta - g_{\alpha\mu}\nabla_\beta\nabla_\upsilon\right)\gamma\left(\Box\right) = \sum_{n=0}^{\infty} \varepsilon^n A_{\mu\upsilon\alpha\beta}^{(n)} \\ &= \left(\eta_{\mu\upsilon}\partial_\alpha\partial_\beta - \eta_{\alpha\mu}\partial_\beta\partial_\upsilon\right)\gamma\left(\Box^{(0)}\right) + \sum_{n=1}^{\infty} \varepsilon^n A_{\mu\upsilon\alpha\beta}^{(n)}, \end{aligned} \tag{14}$$

where

其中

$$\Box^{(0)} = \eta^{\rho\sigma}\partial_\rho\partial_\sigma \tag{15}$$

is the zeroth order expansion of the d'Alembertian operator. We also define and expand the following differential operator, also appearing in (3)

是达朗贝尔算符的零阶展开。我们再对同样出现在 (3) 式中的如下微分算子进行定义和展开:

$$f(\sigma\Box) \equiv (1 + \Box\gamma(\Box)) = \sum_{n=0}^{\infty} \varepsilon^n f^{(n)} = 1 + \Box^{(0)}\gamma\left(\Box^0\right) + \sum_{n=1}^{\infty} \varepsilon^n f^{(n)}$$

$$= f\left(\sigma\Box^{(0)}\right) + \sum_{n=1}^{\infty} \varepsilon^n f^{(n)}. \tag{16}$$

From the definitions and expansions (14) and (16) it follows that

由定义和展开式 (14) 与 (16) 可得:

$$A_{\mu\upsilon\alpha\beta}G^{\alpha\beta} = \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=0}^{n-1} A^{(m)}_{\mu\upsilon\alpha\beta} G^{(n-m)\alpha\beta} = \varepsilon A^{(0)}_{\mu\upsilon\alpha\beta} G^{(1)\alpha\beta}$$

$$+\varepsilon^2 \left(A^{(0)}_{\mu\upsilon\alpha\beta} G^{(2)\alpha\beta} + A^{(1)}_{\mu\upsilon\alpha\beta} G^{(1)\alpha\beta}\right) + \ldots, \tag{17}$$

and

且

$$f(\Box)\, G^{\alpha\beta} = \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=0}^{n-1} f^{(m)} G^{(n-m)\alpha\beta} = \varepsilon f^{(0)} G^{(1)\alpha\beta}$$

$$+\varepsilon^2 \left(f^{(0)} G^{(2)\alpha\beta} + f^{(1)} G^{(1)\alpha\beta}\right) + \ldots. \tag{18}$$

Let us consider the operator Q_2 (Ric) and express it as

现在考虑算符 Q_2 (Ric)，将其表示为:

$$Q_2(\mathrm{Ric}) = \sum_{n_1,n_2=0}^{\infty} c_{n_1,n_2}\sigma\left(D^{n_1}\mathrm{Ric}\right)\left(D^{n_2}\mathrm{Ric}\right) + \sum_{n_1,n_2,n_3=0}^{\infty} c_{n_1,n_2,n_3}\sigma^2\left(D^{n_1}\mathrm{Ric}\right)$$

$$\left(D^{n_2}\mathrm{Ric}\right)\left(D^{n_3}\mathrm{Ric}\right) + \ldots, \tag{19}$$

where Ric is the Ricci scalar or the Ricci curvature, c_{n_1,n_2,n_3} are dimensionless parameters, and the dots indicate a sum of terms of order higher than third in Ric. Moreover, D is a short notation for some operator. For instance, D can represent the contraction of some indices, or it can be a differential operator, e.g., $D = \sigma\Box$. Note that we have omitted indices in (19). We can expand D and Ric in powers of ε as

其中 Ric 是里奇标量或里奇曲率，c_{n_1,n_2,n_3} 是无量纲参数，省略号表示 Ric 中高于三阶的项之和。此外, D 是某个算符的简写。例如, D 可以表示某些指标的缩并, 也可以是一个微分算符, 例如 $D = \sigma\Box$ 。注意，我们在 (19) 式中省略了指标。我们可以将 D 和 Ric 按 ε 的幂次展开为:

$$D = \sum_{n=0}^{\infty} \varepsilon^n D^{(n)}, \ \mathrm{Ric} = \sum_{n=1}^{\infty} \varepsilon^n \mathrm{Ric}^{(n)}, \tag{20}$$

where we have used the fact that $\mathrm{Ric}^{(0)}(\eta) = 0$. From Equation (19) it follows that if $\mathrm{Ric}^{(k)} = 0 \forall k < n$, then one has $\mathrm{Ric} = O(\varepsilon^n)$. Moreover, by means of (20) one has $D = O(1)$, so that

此处我们用到了结论 $\mathrm{Ric}^{(0)}(\eta) = 0$ 。由式 (19) 可得，若 $\mathrm{Ric}^{(k)} = 0 \forall k < n$ ，则有 $\mathrm{Ric} = O(\varepsilon^n)$ 。此外，利用式 (20) 可得 $D = O(1)$ ，因此

$$\mathrm{Ric}^{(k)} = 0 \ \forall k < n \Rightarrow Q_2(\mathrm{Ric}) \sim \varepsilon^{2n}. \tag{21}$$

Finally, the Bianchi identity can be expressed perturbatively by the means of (12) and (13) as

最后，比安基恒等式可以借助式 (12) 和 (13) 表示为微扰形式:

$$0 = \nabla_\alpha G^{\alpha\beta} = \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=0}^{n-1} \nabla_\alpha^{(m)} G^{(n-m)\alpha\beta} = \varepsilon \partial_\alpha G^{(1)\alpha\beta}$$

$$+\varepsilon^2 \left(\partial_\alpha G^{(2)\alpha\beta} + \nabla_\alpha^{(1)} G^{(1)\alpha\beta}\right) + O\left(\varepsilon^3\right). \tag{22}$$

Now we are ready to solve (3) perturbatively. At the lowest order ε , Equation (22) gives $\partial_\alpha G^{(1)\alpha\beta} = 0$, which also implies:

现在我们已经可以对 (3) 进行微扰求解了。最低阶 ε 下，式 (22) 给出 $\partial_\alpha G^{(1)\alpha\beta} = 0$ ，由此还能得到:

$$A^{(0)}_{\mu\upsilon\alpha\beta} G^{(1)\alpha\beta} \propto \left(\eta_{\mu\upsilon}\partial_\alpha\partial_\beta - \eta_{\alpha\mu}\partial_\beta\partial_\upsilon\right)\gamma\left(\Box^{(0)}\right) G^{(1)\alpha\beta} = 0, \tag{23}$$

where we have used the fact that the operator $\gamma\left(\Box^{(0)}\right)$ commute with the ordinary derivatives ∂_α . Therefore, the operator in (17) is null at order ε . Moreover, since $G^{(0)}_{\mu\nu} = 0$ implies $\mathrm{Ric}^{(0)} = 0$ and $\mathrm{Ric} = O(\varepsilon)$, from Equation (21) we see that the term $Q_2(\mathrm{Ric})$ is at least of order ε^2 , hence it does not contribute to the equations at the order ε . In conclusion, in vacuum and at first perturbative order ε , Equation (3) reads

此处我们用到了算符 $\gamma\left(\Box^{(0)}\right)$ 与普通导数 ∂_α 对易的性质。因此，式 (17) 中的算符在 ε 阶为零。此外，由于 $G^{(0)}_{\mu\nu} = 0$ 可推出 $\mathrm{Ric}^{(0)} = 0$ 和 $\mathrm{Ric} = O(\varepsilon)$ ，从式 (21) 可知，项 Q_2 (里奇张量) 至少是 ε^2 阶的，因此它对 ε 阶的方程没有贡献。综上，在真空中一阶微扰 ε 下，式 (3) 可写为

$$f\left(\sigma\Box^{(0)}\right) G^{(1)}_{\mu\nu} = 0. \tag{24}$$

Later on in this section, we will prove that the operator $f\left(\Box^{(0)}\right)$ is invertible. We can use this fact to infer that this equation admits the unique solution

本节后续我们会证明算符 $f\left(\square^{(0)}\right)$ 是可逆的。我们可以利用该结论推出这个方程存在唯一解:

$$G_{\mu\nu}^{(1)}=0 \tag{25}$$

Therefore, our perturbative analysis of (3) at order ε led us to conclude that the first perturbative term $G_{\mu\nu}^{(1)}$ of the Einstein’s tensor must vanish.

因此，我们对式 (3) 在 ε 阶的微扰分析可以得出结论: 爱因斯坦张量的一阶微扰项 $G_{\mu\nu}^{(1)}$ 必须为零。

At second order ε^2 , using the Bianchi identity together with $G_{\mu\nu}^{(1)}=0$, one can show that $\partial_\alpha G^{(2)\alpha\beta}=0$, which in turn gives

二阶 ε^2 下，结合比安基恒等式与 $G_{\mu\nu}^{(1)}=0$ ，可以证明 $\partial_\alpha G^{(2)\alpha\beta}=0$ ，由此可得

$$A_{\mu\nu\alpha\beta}^{(0)}G^{(2)\alpha\beta}\propto\left(\eta_{\mu\nu}\partial_\alpha\partial_\beta-\eta_{\alpha\mu}\partial_\beta\partial_\upsilon\right)\gamma\left(\square^{(0)}\right)G^{(2)\alpha\beta}=0. \tag{26}$$

This equation implies that $A_{\mu\nu\alpha\beta}G^{\alpha\beta}$ vanishes at order ε^2 (see Equation (17)). Moreover, $G_{\mu\nu}^{(1)}=0$ implies that $\mathrm{Ric}^{(1)}=0$, indeed from Equation (21) it also comes that $Q_2\left(\mathrm{Ric}\right)\sim\varepsilon^4$. Thus, at order ε^2 Equation (3) reads

该方程表明 $A_{\mu\nu\alpha\beta}G^{\alpha\beta}$ 在 ε^2 阶为零 (见式 (17))。此外，$G_{\mu\nu}^{(1)}=0$ 可推出 $\mathrm{Ric}^{(1)}=0$ ，实际上从式 (21) 还能得到 $Q_2\left(\mathrm{Ric}\right)\sim\varepsilon^4$ 。因此，在 ε^2 阶下式 (3) 可写为

$$f\left(\sigma\square^{(0)}\right)G_{\mu\nu}^{(2)}=0, \tag{27}$$

which implies that $G_{\mu\nu}^{(2)}=0$, due to the invertibility of $f\left(\sigma\square^{(0)}\right)$.

由于 $f\left(\sigma\square^{(0)}\right)$ 可逆，由此可得 $G_{\mu\nu}^{(2)}=0$

These results are generalized to any order in ε so that, one concludes that Equation (3) implies that $G_{\mu\nu}^{(n)}=0,\forall n\geq 0$. This can be proved by induction showing that, if $G_{\mu\upsilon}^{(m)}=0,\forall m\leq n$, then, Equation (3) implies $G_{\mu\upsilon}^{(n+1)}=0$. Indeed, if $G_{\mu\upsilon}^{(n)}=0$ the Bianchi identity (22) gives $\partial_\alpha G^{(n+1)\alpha\beta}=0$, which implies, by the means of (17), that $A_{\mu\nu\alpha\beta}G^{\alpha\beta}\sim\varepsilon^{n+2}$. Moreover, since $G_{\mu\upsilon}^{(m)}=0,\forall m\leq n$ implies $\mathrm{Ric}^{(m)}=0,\forall m\leq n$, Equation (21) tells us that $Q_2\left(\mathrm{Ric}\right)\sim\varepsilon^{2(n+1)}$. Finally, using (18) we conclude that at the order ε^{n+1} , equation (3) turns into $f\left(\sigma\square^{(0)}\right)G_{\mu\nu}^{(n+1)}=0$, which implies $G_{\mu\nu}^{(n+1)}=0$.

这些结果可推广至 ε 的任意阶，由此可得式 (3) 隐含 $G_{\mu\nu}^{(n)}=0,\forall n\geq 0$ 。这可以通过归纳法证明: 若 $G_{\mu\upsilon}^{(m)}=0,\forall m\leq n$ ，则式 (3) 可推出 $G_{\mu\upsilon}^{(n+1)}=0$ 。事实上，若 $G_{\mu\upsilon}^{(n)}=0$ ，比安基恒等式 (22) 给出 $\partial_\alpha G^{(n+1)\alpha\beta}=0$ ，结合式 (17) 可得 $A_{\mu\nu\alpha\beta}G^{\alpha\beta}\sim\varepsilon^{n+2}$ 。此外，由于 $G_{\mu\upsilon}^{(m)}=0,\forall m\leq n$ 可推出 $\mathrm{Ric}^{(m)}=0,\forall m\leq n$ ，式 (21) 表明 $Q_2\left(\mathrm{Ric}\right)\sim\varepsilon^{2(n+1)}$ 。最后，利用式 (18) 我们可得，在 ε^{n+1} 阶，式 (3) 可化为 $f\left(\sigma\square^{(0)}\right)G_{\mu\nu}^{(n+1)}=0$ ，由此推出 $G_{\mu\nu}^{(n+1)}=0$ 。

Summarizing, we have proved that Equation (3) implies that it must be $G_{\mu\upsilon}^{(n)} = 0$. This, by means of Equation (12), implies that the Einstein's tensor built with the metric (11) must vanish at any perturbative order in ε , namely

综上，我们已经证明式 (3) 表明必须满足 $G_{\mu\upsilon}^{(n)} = 0$。结合式 (12) 可得，由度规 (11) 构造的爱因斯坦张量在 ε 的任意微扰阶都必须为零，即

$$G_{\mu\upsilon}\left(g_{\mu\upsilon}\right) = G_{\mu\upsilon}\left(\eta_{\mu\upsilon} + \varepsilon h_{\mu\upsilon}\right) = \sum_{n=1}^{\infty} \varepsilon^n G_{\mu\upsilon}^{(n)} = 0. \tag{28}$$

Notice that the inverse implication is straightforward, since $G_{\mu\nu} = 0$ implies that (3) is automatically satisfied in vacuum. Thus, equation (3) is satisfied by the metric (11) if and only if the vacuum Einstein equations $G_{\mu\nu}\left(g_{\mu\nu}\right) = 0$ are satisfied. This implies that the dynamics of the small perturbations $h_{\mu\upsilon}$ of the Minkowski spacetime in the nonlocal gravitational model (1) is exactly the same as in Einstein gravity. Indeed, the conditions for the stability of $h_{\mu\upsilon}$ in the model (1) must be the same as in Einstein's gravity. This proves the Theorem 2.

注意到逆推导是直接的: 因为 $G_{\mu\nu} = 0$ 表明式 (3) 在真空中自动成立。因此，度规 (11) 满足式 (3) 当且仅当真空爱因斯坦方程 $G_{\mu\nu}\left(g_{\mu\nu}\right) = 0$ 成立。这说明在非局部引力模型 (1) 中，闵氏时空小扰动 $h_{\mu\upsilon}$ 的动力学与爱因斯坦引力完全一致。因此，模型 (1) 中 $h_{\mu\upsilon}$ 的稳定性条件与爱因斯坦引力完全相同。由此完成了定理 2 的证明。

We can now prove the invertibility of the operator $f\left(\sigma\Box^{(0)}\right)$. This is a consequence of the requirement that $f(z)$ is an entire analytic function with no zeros at finite complex z . We remind that such a requirement is needed in order to ensure that the theory (1) is unitary. Under such hypothesis, we can expand $f\left(\sigma\Box^{(0)}\right)$ in power series of σ , so that the Kernel of such operator will be given by the solutions of the following equation

我们现在可以证明算符 $f\left(\sigma\Box^{(0)}\right)$ 是可逆的。这是 $f(z)$ 为有限复 z 平面上无零点整解析函数这一要求的推论。我们知道，这一要求是保证理论 (1) 幺正性的必要条件。在该假设下，我们可以将 $f\left(\sigma\Box^{(0)}\right)$ 展开为 σ 的幂级数，因此该算符的核由下述方程的解给出

$$f\left(\sigma\Box^{(0)}\right)\phi = \sum_{n=0}^{\infty} c_n \left(\Box^{(0)}\right)^n \sigma^n \phi = 0 \tag{29}$$

with $c_0 \neq 0$. Since (29) is analytic in σ , it must vanish at any order, i.e., it must be $\left(\Box^{(0)}\right)^n \phi = 0, \forall n \geq 0$, which implies that $\phi = 0$. Therefore, the operator $f\left(\sigma\Box^{(0)}\right)$ is invertible because it is linear and its kernel is the zero function. An equivalent proof can be given using Fourier transforms, writing

带有 $c_0 \neq 0$。由于式 (29) 在 σ 内解析，它在任意阶都必须为零，即它必为 $\left(\Box^{(0)}\right)^n \phi = 0, \forall n \geq 0$，这意味着 $\phi = 0$。因此，算子 $f\left(\sigma\Box^{(0)}\right)$ 可逆，因为它是线性的且其核为零函数。我们也可以利用傅里叶变换给出等价证明，写作

$$f\left(\sigma\Box^{(0)}\right)\phi(x) = \int \frac{d^4 k}{(2\pi)^4} f\left(-\sigma k^2\right) \widetilde{\phi}(k)\, e^{ikx}, \tag{30}$$

where $\widetilde{\phi}(k)$ is the Fourier transform of $\phi(x)$. Since $f(z)$ has no zeros for finite z , one has $f\left(-\sigma k^2\right) \neq 0, \forall k^2 < \infty$, and the equation $f\left(\sigma\Box^{(0)}\right)\phi(x) = 0$ has the only solution $\widetilde{\phi}(k) = 0$, that is $\phi(x) = 0$.

其中 $\widetilde{\phi}(k)$ 是 $\phi(x)$ 的傅里叶变换。由于 $f(z)$ 在有限 z 处没有零点，可得 $f\left(-\sigma k^2\right) \neq 0, \forall k^2 < \infty$ ，且方程 $f\left(\sigma\Box^{(0)}\right)\phi(x) = 0$ 只有唯一解 $\widetilde{\phi}(k) = 0$ ，即 $\phi(x) = 0$ 。

We stress once more that the invertibility of the function $f(z)$ is crucial for our result. In order to better understand this point, let us consider a function $f(z)$ with a root of order $q \in \mathcal{N}$ in $z = m^2$. Equation (24) becomes

我们再次强调，函数 $f(z)$ 的可逆性对我们的结论至关重要。为了更好地理解这一点，我们考虑一个在 $z = m^2$ 处存在 $q \in \mathcal{N}$ 阶根的函数 $f(z)$ 。式 (24) 变为

$$f\left(\sigma\Box^{(0)}\right)G^{(1)}_{\mu\nu} = \left[\sum_{n=0}^{\infty} c_n\left(\sigma\Box^{(0)}\right)^n\right]\left(\Box^{(0)} - m^2\right)^q G^{(1)}_{\mu\nu} = 0, \tag{31}$$

which has non null solutions corresponding to the solutions of the equation $\left(\Box^{(0)} - m^2\right)^q G^{(1)}_{\mu\nu} = 0$. Therefore, if the function $f\left(\sigma\Box^{(0)}\right)$ were non invertible, we would have solutions with $G^{(1)}_{\mu\nu} \neq 0$, and the implication in (25) would not be valid.

该方程存在非零解，对应方程 $\left(\Box^{(0)} - m^2\right)^q G^{(1)}_{\mu\nu} = 0$ 的解。因此，如果函数 $f\left(\sigma\Box^{(0)}\right)$ 不可逆，我们会得到存在 $G^{(1)}_{\mu\nu} \neq 0$ 的解，式 (25) 中的推导就不成立了。

We can now discuss the validity of the perturbative approach used in the derivation of Equation (28). This is based on the assumption that the metric tensor, and all the tensors constructed with it, can be expanded in the quantity $\left|\varepsilon h_{\mu\nu}\right| \ll 1$, and that this expansion remains valid after the initial time. The GSA assumption implies that, at the initial time, the metric (11) must satisfy the condition (10) for a sufficiently small μ , which implies that ε must be sufficiently small. Thus, we can take ε small enough in order to guarantee that (10) is satisfied and all the series expansions in ε are convergent at the initial time. As a consequence, equation (28) will be valid in a neighborhood of the initial time, where the evolution of $h_{\mu\nu}$ in the model (1) is the same as in general relativity. This implies that the small perturbations cannot grow too much and break the perturbative expansion in ε , otherwise they would be unstable in Einstein gravity. Indeed, our perturbative scheme must always be valid after the initial time. We also mention that in [6] it has been shown that, in the harmonic gauge and for asymptotically flat initial data satisfying the global smallness assumption, the solutions of the vacuum Einstein equations converge asymptotically in time to Minkowski spacetime; and more precisely it has been shown that $\left|h_{\mu\nu}\right| \lesssim t^{-1}\ln(t)$ converges asymptotically to zero. We also stress that the initial asymptotic flatness condition implies that the initial perturbation $h_{\mu\nu}\left(t_0, \vec{x}\right)$ is thickened into a certain volume, and far from that region, i.e., for $|\vec{x}| \to \infty$, one has $h_{\mu\nu}\left(t_0, \vec{x}\right) \to 0$. On the other hand, the asymptotic behavior $h_{\mu\nu} \sim t^{-1}\ln(t)$ for large times means that the perturbation $h_{\mu\nu}$ will be dynamically spread in all the space, to end up with the Minkowski metric. This supports our conclusion that all the series expansions in $\left|\varepsilon h_{\mu\nu}\right|$ converge, since they are convergent at the initial time.

我们现在可以讨论推导方程 (28) 时所用微扰方法的有效性。该方法基于以下假设: 度规张量以及所有由其构造的张量都可以按 $|\varepsilon h_{\mu\upsilon}| \ll 1$ 展开，且该展开在初始时刻之后仍然有效。GSA 假设表明，在初始时刻，度规 (11) 必须满足条件 (10) 对于足够小的 μ，这意味着 ε 必须足够小。因此，我们可以选取足够小的 ε，以保证 (10) 成立，且所有关于 ε 的级数展开在初始时刻都是收敛的。由此可得，方程 (28) 在初始时刻的邻域内有效，此时模型 (1) 中 $h_{\mu\nu}$ 的演化与广义相对论中的演化一致。这说明小扰动不会过度增长从而破坏关于 ε 的微扰展开，否则它们在爱因斯坦引力中就是不稳定的。实际上，我们的微扰框架在初始时刻之后始终保持有效。我们还要指出，文献 [6] 已经证明，在谐和规范下，对于满足整体小量假设的渐近平坦初始数据，真空爱因斯坦方程的解在时间上渐近收敛于闵氏时空；更准确地说，已经证明 $|h_{\mu\upsilon}| \lesssim t^{-1}\ln(t)$ 渐近收敛于零。我们还要强调，初始渐近平坦条件意味着初始扰动 $h_{\mu\upsilon}(t_0, \vec{x})$ 集中在某个体积内，远离该区域时，即对于 $|\vec{x}| \to \infty$，有 $h_{\mu\upsilon}(t_0, \vec{x}) \to 0$。另一方面，大时间下的渐近行为 $h_{\mu\upsilon} \sim t^{-1}\ln(t)$ 意味着扰动 $h_{\mu\upsilon}$ 会动力学地扩散到整个空间，最终趋近于闵氏度规。这支撑了我们的结论: 既然所有关于 $|\varepsilon h_{\mu\upsilon}|$ 的级数展开在初始时刻收敛，那么它们始终收敛。

We also emphasize that the fact that $Q_2\,(\mathrm{Ric})$ is at least quadratic in the Ricci scalar and Ricci curvature is pivotal in our derivation. To clarify this point, let us consider the following equations of motion:

我们还要强调，$Q_2\,(\mathrm{Ric})$ 至少是里奇标量和里奇曲率的二次项这一点，是我们推导的核心。为阐明这一点，我们考虑如下运动方程:

$$f\,(\sigma\Box)\,G_{\mu\upsilon} + \left(g_{\mu\upsilon}\nabla_\alpha\nabla_\beta - g_{\alpha\mu}\nabla_\beta\nabla_\upsilon\right)\gamma\,(\Box)\,G^{\alpha\beta} = \sigma R_{\mu\tau\rho\sigma}R_\upsilon^{\tau\rho\sigma}, \tag{32}$$

which is (3) with the replacement $Q_{2\mu\upsilon} = -\sigma R_{\mu\tau\rho\sigma}R_\upsilon^{\tau\rho\sigma}$. Terms of this type have not been considered in the previous discussion, since they spoil our result, as discussed below. The perturbative expansion of (32) at the first order in ε is

这是将 (3) 中的 $Q_{2\mu\upsilon} = -\sigma R_{\mu\tau\rho\sigma}R_\upsilon^{\tau\rho\sigma}$ 替换后得到的形式。这类项并未在之前的讨论中涉及，如下文所述，它们会破坏我们的结果。对 (32) 做 ε 一阶微扰展开可得

$$f\left(\sigma\Box^{(0)}\right)G^{(1)}_{\mu\upsilon} = \sigma R^{(1)}_{\mu\tau\rho\sigma}R_\upsilon^{(0)\tau\rho\sigma} + \sigma R^{(0)}_{\mu\tau\rho\sigma}R_\upsilon^{(1)\tau\rho\sigma} \tag{33}$$

which implies $G^{(1)}_{\mu\nu} = 0$ and $R^{(1)}_{\mu\nu} = 0$, since $R^{(0)}_{\alpha\beta\mu\nu} = 0$ by construction. However, there is no reason to expect that $R^{(1)}_{\alpha\beta\mu\nu} = 0$, and in fact the Riemann tensor can be nonzero at order ε, as it is for gravitational waves. At order ε^2 we get

这可以推出 $G^{(1)}_{\mu\nu} = 0$ 和 $R^{(1)}_{\mu\nu} = 0$，因为根据构造有 $R^{(0)}_{\alpha\beta\mu\nu} = 0$。然而，我们没有理由预期 $R^{(1)}_{\alpha\beta\mu\nu} = 0$ 成立，实际上黎曼张量在 ε 阶可以非零，引力波就是这样的情况。在 ε^2 阶我们得到

$$f\left(\sigma\Box^{(0)}\right)G^{(2)}_{\mu\upsilon} = \sigma R^{(2)}_{\mu\tau\rho\sigma}R_\upsilon^{(0)\tau\rho\sigma} + \sigma R^{(0)}_{\mu\tau\rho\sigma}R_\upsilon^{(2)\tau\rho\sigma} + \sigma R^{(1)}_{\mu\tau\rho\sigma}R_\upsilon^{(1)\tau\rho\sigma}$$

$$= \sigma R^{(1)}_{\mu\tau\rho\sigma}R_\upsilon^{(1)\tau\rho\sigma}, \tag{34}$$

which does not imply $G^{(2)}_{\mu\nu} = 0$ because of the term $R^{(1)}_{\mu\tau\rho\sigma}R_\upsilon^{(1)\tau\rho\sigma} \neq 0$. Thus, one does not get Equation (28), and nothing can be said about the evolution of $h_{\mu\upsilon}$ in the case of Equation (34). In conclusion, our argument is valid only if the Riemann tensor enters in $V(\mathcal{R})$ and $Q_{2\mu\nu}$ only through the Ricci scalar and Ricci curvature.

这一点因项 $R^{(1)}_{\mu\tau\rho\sigma}R^{(1)\tau\rho\sigma}_{\nu} \neq 0$ 并不蕴含 $G^{(2)}_{\mu\nu} = 0$ 。因此我们无法得到式 (28)，也就无法对式 (34) 情形下 $h_{\mu\nu}$ 的演化做出任何判断。综上，我们的论证仅在黎曼张量仅通过里奇标量和里奇曲率进入 $V(\mathcal{R})$ 与 $Q_{2\mu\nu}$ 时才成立。

Generalization to Ricci-Flat and Maximally Symmetric Solutions

推广到里奇平坦与极大对称解

The results discussed in the previous sections on the stability of Minkowski spacetime in nonlocal gravity can be generalized to maximally symmetric and Ricci-flat spacetimes, that is, vacuum solutions of the Einstein equations with and without a cosmological constant Λ . In fact, a class of nonlocal gravitational theories can be built in such a way that it encompasses all the Ricci-flat and maximally symmetric solutions of general relativity. Moreover, the dynamics of the small perturbations of such solutions is the same as in Einstein gravity, indeed they are as stable as in general relativity. We will discuss these results, remanding the reader to Reference [9] for all the details.

前面几节讨论的非局域引力中闵氏时空的稳定性结果，可以推广到极大对称时空与里奇平坦时空，即带和不带宇宙学常数的爱因斯坦方程真空解 Λ 。事实上，这类非局域引力理论可以构造为囊括广义相对论所有里奇平坦解与极大对称解的形式。此外，这类解的小扰动动力学与爱因斯坦引力完全一致，其稳定性也和广义相对论中的情况相同。我们将在此讨论这些结果，所有细节请读者参考文献 [9]。

A class of theories that is compatible with unitarity and renormalizability at quantum level and is suitable for our purposes is obtained generalizing (1) as

为满足我们的需求，一个在量子层面满足幺正性和可重整性的理论类，可以通过推广 (1) 得到，形式如下

$$S_{\mathrm{g}} = -\frac{2}{\kappa_D^2}\int d^4x\sqrt{-g}\left[R + \left(E_{\mu\nu} - \frac{1}{2}Eg_{\mu\nu}\right)\gamma(\Delta_\Lambda)E^{\mu\nu} + V(E)\right], \tag{35}$$

where we have defined the tensor $E_{\mu\nu}$

其中我们定义了张量 $E_{\mu\nu}$

$$E_{\mu\nu} \equiv G_{\mu\nu} + \Lambda g_{\mu\nu}, \tag{36}$$

and its trace $E \equiv g^{\mu\nu}E_{\mu\nu}$. The operator Δ_Λ is the Lichnerowicz operator, which, acting on a rank-2 symmetric tensor, gives

以及它的迹 $E \equiv g^{\mu\nu}E_{\mu\nu}$ 。算符 Δ_Λ 是里奇诺维茨算符，它作用在二阶对称张量上得到

$$\Delta_\Lambda X_{\mu\nu} = 2R^\sigma{}_{\mu\nu\tau}X^\tau{}_\sigma + \left(R_{\mu\sigma} - \Lambda g_{\mu\sigma}\right)X^\sigma{}_\nu$$

$$+ \left(R_{\sigma\upsilon} - \Lambda g_{\sigma\upsilon}\right)X^\sigma{}_\mu - \Box X_{\mu\upsilon}. \tag{37}$$

Also, the definition of the form factor in (2) is replaced by

此外，(2) 中的形状因子定义替换为

$$\gamma\left(\Delta_\Lambda\right) = \frac{f\left[-\sigma\left(\Delta_\Lambda + 4\Lambda\right)\right] - 1}{-\left(\Delta_\Lambda + 4\Lambda\right)}, \tag{38}$$

which ensures the quantum unitarity of the theory, provided that $f(z)$ is an entire function without zeros at finite z . Moreover, the generalized potential $V(E)$, is at least cubic in $E_{\mu\upsilon}$ and E .

只要 $f(z)$ 是在有限 z 处没有零点的整函数，该形式就能保证理论的量子幺正性。此外，广义势 $V(E)$ 在 $E_{\mu\upsilon}$ 和 E 中至少是三阶的。

We can generalize the stability Theorem 2 for the class of nonlocal gravitational models (35), stating the following:

我们可以将非局域引力模型 (35) 的稳定性定理 2 推广为如下结论:

Theorem 3. Ricci-flat ($\Lambda = 0$) and maximally symmetric ($\Lambda \neq 0$) spacetimes are vacuum solutions of the nonlocal gravity (35). Moreover, the small perturbations of these solutions satisfy the same equations of motion as in general relativity. Then, if a Ricci-flat or maximally symmetric solution is stable in Einstein gravity, it is also stable in the nonlocal model (35).

定理 3. 里奇平坦 ($\Lambda = 0$) 时空与极大对称 ($\Lambda \neq 0$) 时空是非局域引力 (35) 的真空解。此外，这些解的小扰动满足和广义相对论中相同的运动方程。因此，若一个里奇平坦或极大对称解在爱因斯坦引力中是稳定的，那么它在非局域模型 (35) 中也是稳定的。

Let us give a sketch of the proof of this theorem. The equations of motion resulting from the variation of the action (35) are

下面我们给出该定理的证明概要。对作用量 (35) 变分得到的运动方程为

$$f\left[-\sigma\left(\Delta_\Lambda + 4\Lambda\right)\right]E_{\mu\upsilon} + \left(g_{\mu\upsilon}\nabla_\alpha\nabla_\beta - g_{\alpha\mu}\nabla_\beta\nabla_\upsilon\right)\gamma\left(\Delta_\Lambda\right)E^{\alpha\beta} + Q_{2\mu\upsilon}(\mathrm{E})$$

$$= 8\pi G_{\mathrm{N}} T_{\mu\upsilon} \tag{39}$$

where $Q_{2\mu\upsilon}(\mathrm{E})$ includes all the terms at least quadratic in $E_{\mu\upsilon}$ and E . Let us consider a Ricci-flat or maximally symmetric spacetime $g^{(0)}_{\mu\upsilon}$, such that

其中 $Q_{2\mu\upsilon}(\mathrm{E})$ 包含了所有在 $E_{\mu\upsilon}$ 和 E 中至少二次的项。我们考虑一个里奇平坦或极大对称时空 $g^{(0)}_{\mu\upsilon}$ ，满足

$$E_{\mu\upsilon}^{(0)} \equiv E_{\mu\upsilon}\left(g^{(0)}\right) = G_{\mu\upsilon}^{(0)} + \Lambda g_{\mu\upsilon}^{(0)} = 0, \tag{40}$$

where $G_{\mu\nu}^{(0)}$ is the Einstein tensor constructed with the metric $g_{\mu\nu}^{(0)}$. The case of Ricci-flat solutions corresponds to $\Lambda = 0$, while maximally symmetric solutions have $\Lambda \neq 0$. From (40) it is evident that the metric tensors $g_{\mu\nu}^{(0)}$, for which $E_{\mu\nu} = 0$, are exact vacuum solutions of (39).

其中 $G_{\mu\nu}^{(0)}$ 是由度规 $g_{\mu\nu}^{(0)}$ 构造的爱因斯坦张量。里奇平坦解对应 $\Lambda = 0$ ，而极大对称解满足 $\Lambda \neq 0$ 。从 (40) 可以明显看出，满足 $E_{\mu\nu} = 0$ 的度规张量 $g_{\mu\nu}^{(0)}$ 是 (39) 的精确真空解。

Thus, we can study the dynamics of the small perturbations of $g_{\mu\nu}^{(0)}$. We consider perturbations around the background $g_{\mu\nu}^{(0)}$, expanding the metric tensor $g_{\mu\nu}$ in powers of a small parameter $\varepsilon \ll 1$ as

因此，我们可以研究 $g_{\mu\upsilon}^{(0)}$ 的小扰动动力学。我们考虑背景 $g_{\mu\upsilon}^{(0)}$ 附近的扰动，将度规张量 $g_{\mu\nu}$ 按小参数 $\varepsilon \ll 1$ 展开为

$$g_{\mu\upsilon} = \sum_{n=0}^{\infty} \varepsilon^n h_{\mu\upsilon}^{(n)}, \ h_{\mu\upsilon}^{(0)} \equiv g_{\mu\upsilon}^{(0)}. \tag{41}$$

By means of (41), we can expand all the relevant quantities in powers of ε . For instance, for the Einstein tensor and the tensor $E_{\mu\upsilon}$ one has

利用 (41)，我们可以将所有相关物理量按 ε 幂次展开。例如，对爱因斯坦张量和张量 $E_{\mu\upsilon}$ 有

$$G_{\mu\upsilon}\left(g_{\mu\upsilon}\right) = \sum_{n=0}^{\infty} \varepsilon^n G_{\mu\upsilon}^{(n)}, \ \text{ with } \ G_{\mu\upsilon}^{(0)} = -\Lambda g_{\mu\upsilon}^{(0)},$$

(42)

$$E_{\mu\upsilon}\left(g_{\mu\upsilon}\right) = \sum_{n=0}^{\infty} \varepsilon^n E_{\mu\upsilon}^{(n)} = 0, \ E_{\mu\upsilon}^{(0)} = 0.$$

Moreover, with some algebra, and setting $T_{\mu\nu} = 0$, equation (39) can be recast in the form

此外，经过代数整理并令 $T_{\mu\nu} = 0$ 后，方程 (39) 可以改写为如下形式

$$f\left[-\sigma\left(\Delta_\Lambda + 4\Lambda\right)\right] E_{\mu\upsilon} + O\left(\mathbf{E}^2\right) = 0. \tag{43}$$

where $f\left[-\sigma\left(\Delta_\Lambda + 4\Lambda\right)\right]$ is an invertible operator.

其中 $f\left[-\sigma\left(\Delta_\Lambda + 4\Lambda\right)\right]$ 是可逆算符。

At this point, and in the spirit of the proof of Theorem 2, one can replace the second of the equations (42) in (43) and find a recursive relation for the coefficients $E_{\mu\upsilon}^{(n)}$ which implies that, if $E_{\mu\upsilon}^{(0)} = 0$, then $E_{\mu\upsilon}^{(n)} = 0$ for any n . That means that, the tensor $E_{\mu\upsilon}$ must be zero at any perturbative order in ε , which implies that

至此，遵循定理 2 的证明思路，我们可以将 (43) 中的方程 (42) 的第二个方程替换，得到系数 $E_{\mu\nu}^{(n)}$ 的递推关系，该关系表明: 若 $E_{\mu\nu}^{(0)}=0$，则对任意 n 都有 $E_{\mu\nu}^{(n)}=0$ 成立。这意味着，张量 $E_{\mu\nu}$ 在 ε 的任意微扰阶都必须为零，由此可得

$$E_{\mu\nu}\left(g_{\mu\nu}\right)=G_{\mu\nu}+\Lambda g_{\mu\nu}=0. \tag{44}$$

Equation (44), together with Equation (39), implies that, if the perturbed metric (41) is a solution of the nonlocal equations of motion (39) in vacuum, it is also a solution of the Einstein equations in vacuum, and vice versa. That means that the dynamics of the small perturbations $h_{\mu\nu}$ in nonlocal gravity is the same as in general relativity. Indeed, a vacuum solution of the theory (35) is stable, if it is a stable solution of general relativity. This proves Theorem 3.

方程 (44) 与方程 (39) 共同表明: 若受扰度规 (41) 是真空中非局部运动方程 (39) 的解，那么它也必然是真空中爱因斯坦方程的解，反之亦然。这说明非局部引力中小微扰 $h_{\mu\nu}$ 的动力学与广义相对论中的完全一致。事实上，理论 (35) 的真空解若为广义相对论中的稳定解，那么它本身就是稳定的。由此完成了定理 3 的证明。

For completeness, we mention that first results on the stability of black hole solutions in nonlocal gravity have been obtained in [10]. However, such studies considered only the linear stability, while Theorem 3 is valid at any perturbative order. We also mention that this class of stability theorems can be extended to the case of a nonlocal theory in presence of matter fields. This result has been obtained in [11] and will be discussed elsewhere in this book.

为完备起见，我们提及: 非局部引力中黑洞解稳定性的首个结果已在文献 [10] 中得到。不过这类研究仅讨论了线性稳定性，而定理 3 在任意微扰阶都成立。我们还指出，这类稳定性定理可以推广到存在物质场的非局部理论情形，该结果已在文献 [11] 中得到，本书将在其他部分对此进行讨论。

Cross-References

交叉引用

Classical and Quantum Nonlocal Gravity

经典与量子非局域引力

- Gauge Invariant Renormalizability of Quantum Gravity

- 量子引力的规范不变可重整性

Nonlocal Gauge Theories Including Quantum Gravity

包含量子引力的非局域规范理论

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